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Optimum Structural Design with Plate Bending Elements—A Survey

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Introduction

THE large number of papers published recently in the field of optimal structural design motivated the writing of several good reviews of the state of the art (e.g., Refs. 1 and 2). However, because of the large number of papers that need to be covered in these reviews (two hundred references in Ref. 1, for example), it is difficult to treat adequately all aspects of research in this field. The solution to this problem is specialized reviews that focus on one aspect of optimum structural design. Past reviews of this type concentrated on such aspects as optimization techniques (e.g., Refs. 3-6) or types of constraints (e.g., Refs. 7-10). The present paper is concerned with the structural optimization of a particular structural element — the plate.

While most of the work reviewed here deals with the minimum weight design of plates subject to various constraints, some papers deal with the dual problem. For example, some papers solve the problem of maximizing the fundamental frequency of a plate at a given weight or volume. This problem is equivalent to the minimum weight design of the plate with a prescribed frequency. No attempt is made, therefore, to distinguish the two types of approaches and they are both regarded here as solutions to the minimum weight design problem.

Because of the availability of ever more powerful computers, the trend in the optimum design of plates is away from methods that are tailored to the specific geometry and loads and toward methods that can easily be programmed for any kind of plate. For the analysis part this trend manifests itself in the use of finite element or similar methods. For the optimization part, the trend is away from variational techniques and more towards the use of general numerical optimization algorithms. Therefore, the present review is not comprehensive, but tends to emphasize the discussion of finite element, numerical optimization type developments. Because of this emphasis, the discussion is limited to papers published in the last two decades.

In reviewing some of the work on optimum design of plates, one notes the radically different designs obtained for the same problems by different researchers. This is related to the fact that the optimum design often has a discontinuous thickness distribution (e.g., Ref. 11), and is achieved through a plate reinforced by ribs. Many investigators assume a continuous plate design and, therefore, their results are of questionable validity.

Stress and Displacement Constraints

Fully Stressed Design and Optimality Criteria

Early work in designing plates for stress constraints adopted the fully stressed design (FSD) approach. That is, each point in the plate is assumed to reach the allowable stress at the extreme fibers. While this criterion may not lead to a truly optimal design for every plate geometry and loading combination it is probably adequate for the simple geometries that were considered. For example, Lee[12-14] used a constant radial stress criterion for the design of circular plates. For the case of a simply supported circular plate subject to uniform loads and an annular plate subject to constant moments at its inner and outer peripheries, he was able to solve the equation of equilibrium and obtain an analytical expression for the optimal thickness distribution.

It is possible also to obtain rigorous optimality criteria for stress constraints. Banichuk,[15] for example, obtains the optimality criterion for optimizing hole shapes in a plate subject to bending.

When the geometry of a plate is irregular, the stresses and displacements have to be obtained numerically, most commonly by the finite element method. The great advantage of the finite element method is that it can treat without additional effort anisotropy, irregular geometry and nonuniform thickness distribution. This advantage applies also for optimization problems.

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One way of adding optimization capabilities to a finite element program is to add a simple resizing scheme based on the fully stressed design approach or a more rigorous optimality criterion. For plate elements with inplane loads and subject to only stress constraints, the stress-ratio technique has been very popular. The stress-ratio technique prescribes the resizing of a plate element thickness by multiplying it by the stress ratio in the element. That is, the resized thickness t_{new} is related to the current thickness t_{old} as

$$t_{new} = t_{old}(\sigma/\sigma_{al})$$

where σ is a measure of the stress intensity in the element (e.g., Von Mises equivalent stress) and σ_{al} is the allowable value for this measure of the stress intensity. It is clear that this approach is based on the assumption that the stress resultants in the element are independent of its thickness. Because, in general, this is not exactly true, we have to perform several iterations to achieve convergence and there is no guarantee that the resultant design is optimal. However, convergence is usually fast and seems to typically result in optimal or near optimal designs. For plate elements subject to bending loads the stress ratio technique may be generalized based on the same assumption that stress resultants are independent of the element thickness. Rao[16] describes such a generalization for the case of bending with no inplane loads based on a limit on the bending stress in the direction of the maximum bending moment. The resized thickness is given as

$$t_{new} = \sqrt{6M/\sigma_{al}}$$

where M is the maximum bending moment in the element. Kiusalaas and Reddy[17] present a more general technique applicable also for combination of membrane and bending stresses based on a modified Von Mises yield criterion which they incorporated in their optimization program DESAP-1. Their technique requires the solution of a quartic equation for the resized thickness.

When displacement constraints are present the fully stressed design approach is no longer applicable. It is possible, however, to obtain the optimality conditions for the plate and solve them directly. This can be done easily when a very simple thickness distribution for the plate is assumed. Thus, Sherman[18,19] assumed a two parameter exponential distribution for the thickness of a circular plate under normal pressure and subject to stress and displacement constraints and obtained a series solution. Banichuk[20] obtained the general optimality conditions for plates of maximum stiffness for a point load. He then obtained analytical solutions for some examples. In more complicated cases, a numerical solution may be called for solving the optimality equations. Huang[21] obtained an iterative numerical solution for the design of a circular plate of maximum stiffness under uniform pressure using finite differences. Similarly, Erbatur and Mengi[22-24] used a finite difference iterative numerical method for solving the optimality equations of circular plates under various load conditions subject to stress and displacement constraints.

The success of the stress-ratio resizing applied in conjunction with the fully stressed design approach has led to development of simple resizing formulas for satisfying optimality criteria for many constraints (see, for example, Venkayya[25]). The most successful applications of these have been to design for a single displacement or stiffness constraint. For plate bending problems, Armand and Lodier[26] developed an optimality criterion and a simple resizing technique for a single displacement constraint with constant-moment plate-bending triangular elements. The procedure is applied to obtain the minimum mass design of simply supported and clamped square plates for concentrated and distributed loads. However, for more general and multiple constraints, simple resizing schemes based on optimality criteria may not be adequate. In such cases, it is necessary to use more general mathematical programming techniques (including generalized optimality criteria) which require computation of derivatives of constraints with respect to design variables.

Mathematical Programming

The application of mathematical programming techniques to structural design carries with it the advantage that several optimization algorithms are available as computer subroutines which can be used for a wide variety of problems. The designer has to couple such a subroutine to a problem dependent code which supplies it with values of the constraints and their derivatives. He is, however, relieved of the need to write the optimization code himself. For structures modeled by finite elements, there are three alternate ways of obtaining constraint derivatives. One way is to obtain the derivatives directly by repeating the analysis with perturbed values of the design variables and using a finite difference expression for the desired derivatives. This approach has accuracy problems and is also quite a bit more expensive computationally than the remaining two approaches for obtaining derivatives of stresses and displacements under static loads. A second approach is to obtain expressions for the solution for the desired derivatives from the continuum equations of the plate and then use the finite element solution for obtaining numerical values. This approach was used by Haug and Feng[27,28] for designing dynamically loaded plates subject to displacement constraints with the gradient projection algorithm. These first two approaches have the advantage that they may be applied with a "black box" finite element code independently of the type of finite elements used to model the plate.

The third approach for obtaining derivatives is to directly differentiate the finite element equations. The required derivatives of displacement or stress components are then calculated from the differentiated equations either directly or through the use of adjoint variables (or "dummy loads"). This approach requires the derivatives of the element stiffness matrices and is therefore code dependent. However, once implemented in a particular finite element program, it is more problem independent than the second approach. This last approach was used by the authors in the PARS program.[29] PARS is a companion code to the SPAR[30] general-purpose finite element program and it contains an optimization module based on an extended interior penalty function technique. The plate elements in SPAR are hybrid stress elements developed by Pian.[31] PARS has been applied to a variety of plate problems[32-34] including both membrane and bending effects subject to static stress and displacement constraints.

Rao[35] also used the third approach for obtaining analytical derivatives of element stiffness matrices for the design of a rectangular plate. The plate carries two alternative systems of loads, is simply supported on two opposite sides and is free on the remaining two sides. The plate is designed subject to stress constraints using the gradient projection algorithm.

For stress and displacement constraints in the elastic range the use of the finite element method seems to go hand in hand with the use of ready made mathematical programming algorithms. The application of both methods is clearly helped by the continuing decline in the cost of computation provided by digital computers. However, there are many applications of mathematical programming algorithms to plate problems which are solved analytically or through numerical techniques other than finite element. Aristov and Trotskii,[36] for example, use numerical optimum control techniques to obtain the design of an annular plate under general normal loads subject to constraints on the maximum tangential and radial bending stresses. Alspaugh and Huang[37] use similar techniques for the design of a circular-sandwich plate subject to stress and displacement constraints. Banichuk and coworkers[38,39] use the gradient projection optimization algorithm combined with a variational approach to design square plates subject to stiffness constraints. Kirsch[40] applies linear programming techniques to the optimum choice of prestressing loads for designing prestressed plates. Additional examples are given in the next section. There are also applications where the finite element method is used but the optimization is not automated but carried on by exhaustive parameter studies (e.g., Ref. 41).

Smooth vs Ribbed Designs

It appears that for most problems that include a stiffness constraint, the optimum design does not correspond to a smooth thickness distribution but to a stiffened or ribbed design. Thus, Simitses[42] obtained a much stiffer design than Huang's[21] for a circular plate under normal pressure by using stiffeners. Reiss[43] showed that the optimality conditions that are typically used for plate design under stiffness constraints are merely stationarity conditions. He also showed that one may obtain arbitrarily stiff designs for a given weight. Practically, of course, this is not realizable because thin plate theory may not be used for ribbed designs. Instead, the design problem has to be reformulated as a stiffened plate problem. Stress, displacement and stability constraints will then limit the height of the stiffeners and probably result in a maximum value of the stiffness for a given weight. It is clear, therefore, that the optimum designs obtained in some of the previously discussed works are not really optimal.

Design Against Plastic Collapse

Because the equations of compatibility can be disregarded in designing a structure against plastic collapse, this design problem is easier than the design of the same structure subject to stress and displacement constraints in the elastic range. It is, therefore, not surprising that there have been more papers published in the 1950's and 1960's on the optimal design of plates against plastic collapse than on design in the elastic range. Most of these papers[44-50] were applied to sandwich designs with constant core and variable face sheet thickness. They were based on the theorem of constant energy dissipation due to Drucker and Shield[51] which can be summarized as follows: "A necessary condition for the attainment of minimum volume is that the plate be at incipient plastic flow and dissipate energy at constant rate at all points of its faces." The results were obtained analytically and were limited to simple geometric. Shield,[44] for example, has obtained an analytical solution for the minimum weight design of transversely loaded elliptical sandwich plates with either simply supported or clamped edge conditions.

Results were also obtained for full-thickness circular plates by Brotchie[52] and Marcal.[53] Brotchie obtained his design by using a simplistic failure criterion requiring that both radial and tangential bending moments be at their plastic limit everywhere. Marcal obtained the design for plastic collapse of the plate by means of an associated nonlinearly elastic plate.

Because analytical methods are suitable only for simple geometries, we see in this area too a transition to mathematical programming methods of optimization and finite element methods of analysis. Chan[54] shows that some of the old methods of optimal design correspond to various formulations of mathematical programming. If perfectly plastic material behavior is assumed, many plastic collapse problems can be treated as linear programming problems. This approach has been used extensively in the limit design of trusses and frames (see Ref. 55 for references) and has been also extended to plane stress problems.[56] Plastic design of plates and shells via linear programming was introduced by Rzhanitsyn[57] and Borkowski (previously Borkausakas) and Chyras.[58] Borkowski also applied linear programming in conjunction with finite element analysis to the problem of optimum reinforcement of plates.[59] Lamblin et al.[60] applied a similar technique to the design of circular plates.

Buckling Constraints

Unlike the case of stiffness constraints, most papers on the design of plates subject to buckling constraints recognize the fact that the optimum design is a stiffened plate. One of the few papers that considered the design of an unstiffened plate against buckling is Frauenthal's work on circular plates.[61] However, Simitses[42] showed that much higher buckling loads may be obtained for the same weight by using stiffened configurations. Another work which is concerned with unstiffened plates is Rammerstorfer's,[62] which considered optimal choice of initial stresses for raising buckling loads and vibration frequencies. Most of the work on optimum design of plates against buckling does start from a stiffened configuration so that stiffeners have to be designed simultaneously with the plate. As a result, several buckling modes have to be considered including global buckling, local buckling between stiffeners, and stiffener buckling. For some simple problems it is possible to obtain the optimum design without resorting to any optimization algorithm. Libai[63] did that for the design of a uniform thickness plate stiffened with a single blade stiffener under axial compression. Only three design variables were used: the plate thickness, the blade thickness and height.

Early attempts at obtaining optimum designs of stiffened plates were based on the concept of two or more modes of failure being critical simultaneously for the applied load. Simple expressions were used to approximate the failure load of the stiffened plate in the various modes. It was therefore possible to obtain designs that satisfied the simultaneous failure criterion without resorting to any high powered mathematical or numerical tools. A representative of such an approach which also provides references to previous work of this type is the paper by Crawford and Burns.[64] A recent application of the same approach to the design of channel columns is provided by Yoshida.[65] In Yoshida's work, the ultimate strength of the plate elements is matched to the overall buckling loads. Schmit et al.[66] introduced a special purpose mathematical programming algorithm based on the steepest descent technique as a more rigorous alternative to the simultaneous failure approach. However, the computational expense of using early mathematical programming algorithms coupled with the reasonable results obtained from the simultaneous failure approach discouraged any further work in that direction for several years.

The use of composite materials gave a new push to the use of mathematical programming optimization methods for stiffened plate designs. This was due to the increased complexity of the design and the attendant inadequacy of the simultaneous failure approach. Also, by the early 1970's, several standard, more efficient mathematical programming algorithms were available. Agarwal and Davis[67] obtained designs for hat stiffened composite panels under uniaxial compression using the AESOP[68] optimization package. Stroud and Agranoff[69] also employed AESOP to design such panels for combined loads. Sobieski[70] obtained design of a composite aircraft wing as an assemblage of sandwich panels subject to stress and buckling constraints, using the CONMIN[71] optimization package.

The work discussed above still employed simple approximate expressions for calculating buckling loads. Later work by Stroud et al.[72,73] introduced a rigorous buckling analysis based on the VIPASA program.[74] VIPASA treats the stiffened plate as an assemblage of smaller plates and is therefore similar in concept to a finite element code. A trigonometric solution is assumed along each plate length and an exact solution to the resulting ordinary differential equation is obtained in the other direction. The AESOP package used in the earlier work was replaced by the CONMIN[71] optimization package which is based on the feasible directions algorithm. The stiffened panel resizing code of Refs. 72 and 73 is called PASCO.

Some work has also been done on designing minimum weight unstiffened composite plates subject to buckling constraints and using standard mathematical programming algorithms. Waddoups and his co-workers[75] used an interior penalty approach while Schmit and Farshi[76] used the inscribed hyperspheres method. Hirano[77] obtained the design of a composite plate under axial compression for maximum buckling load using Powell's algorithm.

Another problem that is often encountered in designing plates subject to buckling constraints is that the plate is a small element in a more complex structure which has to be designed subject to some global constraints. This problem is very common in aerospace structures such as wing or fuselage structures. The problem which is faced by the designer is how to interface the local design of each panel with the global design of the entire structure. Typically, a simplified buckling analysis is used for the local design of the plates. Several approaches to the problem have been used. Sobieski (previously Sobieszczanski) and Loendorf[78] used mathematical programming at the local design level and a fully-stressed-design stress-ratio technique for the global resizing. Schmit and Ramanathan[79] used a multilevel approach whereby the local level design was accomplished subject to a constant stiffness requirement so as to minimize the interaction with the global design. Starnes and Haftka[80] took the more rigorous and more expensive approach of treating all variables as global variables. All three papers use finite elements for the analysis and standard mathematical programming algorithms.

Vibration Constraints

The problem of designing a plate subject to a frequency constraint is very similar to the problem of designing it subject to a buckling constraint. In fact, the PASCO code[73] discussed in the previous section may be used for designing either for buckling or frequency constraints. However, for design subject to buckling constraints it was recognized a long time ago that rib stiffened plates are more efficient ones than unstiffened ones. For design subject to frequency constraints this was recognized only recently (e.g., Refs. 11 and 81), and most of the research was directed to unstiffened configurations. In fact, the debate on this point is still going on. Grinev and Filipov[82] argued that upper bounds on the thickness are needed for an optimum to exist while Seiranyan[83] showed that the continuous solution are strong minima if a discontinuous solution is excluded. However, discontinuous or ribbed designs are acceptable in practice, and so many of the unstiffened, optimum designs may be of little value practically.

The design of plates subject to frequency constraints is often so difficult that it is not possible to obtain analytical solutions. Even the design of a circular plate with an axisymmetric vibration mode, a one dimensional problem, had to be solved numerically by Olhoff.[84] Armand[85,86] obtained analytically the minimum weight design of a simply rectangular shear plate with a prescribed fundamental frequency. Banichuk and Mironov[87] obtained an analytical solution for the design of sandwich plates oscillating in an ideal fluid. These problems however are equivalent to designing a membrane with no bending stiffness.

The numerical solution of plate design subject to frequency constraints can be specifically tailored to the problem at hand or use can be made of general mathematical programming techniques and finite element analysis. Olhoff[84,88] employs a variational approach using the Rayleigh quotient to the general equations of thin elastic plates, for obtaining the minimum weight designs of circular and rectangular plates. A similar approach was employed by Bert[89,90] for a simply supported laminated plate and by Banichuk[91] for a plate immersed in an ideal fluid. Gura and Seyranian[92,93] combined the optimal solutions for a circular plate subject to a frequency constraint, a buckling constraint, and a deflection constraint to construct a quasioptimal solution for the design of a plate subject to all three constraints.

Other researchers tend more to the use of general purpose techniques. Haug and co-workers[94-96] used a generalized steepest descent method for the design of a simply supported rectangular plate. The vibration analysis was based on the equations for thick plates using a collocation technique for the solution. Katarya[97] obtained the minimum weight design of a rectangular wing subject to strength and frequency constraints. The design variables were the dimensions of the chord and semispan and the thickness of the spars and the wing skin. The wing was analyzed using a cylindrical tube theory, and the Fiacco-McCormick interior penalty method was used for optimization. Rand and Shen[98] obtained the optimum design of a composite cylindrical shell using finite elements and a penalty function optimization technique coupled with the Fletcher-Powell quasi-Newton algorithm. A similar approach was used by Rao and Singh[99] for the design of laminated plates. Oluyomi and Tabrok[100] used an optimality criterion approach coupled with finite element analysis for the design of a pentagonal plate.

Other Constraints

The list of constraints discussed in the previous sections is by no means complete. For example, constraints on panel flutter (e.g., Refs. 101-109), crack growth (e.g., Ref. 110) or safety[111] have not been discussed. These constraints were considered to be of very specialized nature and omitted for this reason.

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